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Fundamental groups of the complements of certain plane non-tame torus sextics

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Abstract

The moduli space of torus sextics with the configuration of singularities $\{A_2 + A_5 + 2E_6\}$ has two connected components. We compute the fundamental groups $\pi_1(\mathbb{CP}^2 - C)$ for sextics C in both components and study their differences.

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1. Introduction and statement of the result

A sextic $\{(X:Y:Z) \in \mathbb{CP}^2;\ F(X,Y,Z) = 0\}$ is said of *torus type* if there is an expression $F(X,Y,Z) = F_2(X,Y,Z)^3 + F_3(X,Y,Z)^2$, where F_2 and F_3 are homogeneous polynomials of degree 2 and 3 respectively. A sextic of torus type $\{(X:Y:Z) \in \mathbb{CP}^2;\ F_2(X,Y,Z)^3 + F_3(X,Y,Z)^2 = 0\}$ is said *tame* if its singularities are sitting only at the intersection of the conic and the cubic defined by $F_2(X,Y,Z) = 0$ and $F_3(X,Y,Z) = 0$ respectively. In [18], Pho classified the configurations of singularities on tame torus sextics. In [16], Oka and Pho computed the fundamental group $\pi_1(\mathbb{CP}^2 - C)$ for any irreducible

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tame torus sextic C. In [17,13], they classified the configurations of singularities on non-tame torus sextics. In general, in the case of non-tame torus curves, the computation of the fundamental group is more complicated. Some examples are given in [16,2]. In this paper, we give a new family of examples. More precisely, we compute the fundamental group $\pi_1(\mathbb{CP}^2 - C)$ for any irreducible non-tame torus sextic C with the following configuration of singularities $\{A_2 + A_5 + 2E_6\}$ (in fact, by [18,17,13], a torus sextic C having this configuration of singularities is necessarily irreducible and non-tame).

Let $(X:Y:Z)$ be homogeneous coordinates on $\mathbb{CP}^2$, and let (x, y) be the affine coordinates defined by $x := X/Z$ and $y := Y/Z$ on $\mathbb{CP}^2 - \{Z = 0\}$. Denote by $\mathcal{M}$ the moduli space of (irreducible non-tame) torus sextics in $\mathbb{CP}^2$ with the configuration of singularities $\{A_2 + A_5 + 2E_6\}$. By [17], $\mathcal{M}$ has two irreducible components $\mathcal{M}_1$ and $\mathcal{M}_2$ which consist of the PGL(3, $\mathbb{C}$)-orbits of the curves C_1 and C_2 given by the affine equations $f_1(x, y) = 0$ and $f_2(x, y) = 0$, where

$$\begin{aligned} f_1(x,y) := {}& \bigl(y^2 - 1 - 5x^2 + 4x^2\sqrt{3} + 6x - 4x\sqrt{3}\bigr)^3 \\ & + \frac{36x^2}{-2\sqrt{3}+3}\bigl(5y^2 - 3y^2\sqrt{3} - 9x^2 + 5x^2\sqrt{3} \\ & + 14x - 8x\sqrt{3} - 5 + 3\sqrt{3}\bigr)^2 \end{aligned}$$

and

$$\begin{aligned} f_2(x,y) := {}& \bigl(y^2 - 1 - 5x^2 - 4x^2\sqrt{3} + 6x + 4x\sqrt{3}\bigr)^3 \\ & + \frac{36x^2}{2\sqrt{3}+3}\bigl(5y^2 + 3y^2\sqrt{3} - 9x^2 - 5x^2\sqrt{3} \\ & + 14x + 8x\sqrt{3} - 5 - 3\sqrt{3}\bigr)^2. \end{aligned}$$

Theorem. *Let C be a curve in $\mathcal{M}$. Then, the fundamental group $\pi_1(\mathbb{CP}^2 - C)$ is presented by generators and relations as follows*:

$$\pi_1\bigl(\mathbb{CP}^2 - C\bigr) = \begin{cases} \langle a, b \mid (aba)^2 = (bab)^2,\ bab^3 = a^3ba, \\ \qquad (ab)^3 = e\rangle, \quad \text{for } C \in \mathcal{M}_1, \\ \langle a, b, c \mid aba = bab,\ cacb = acba, \\ \qquad bacb = acbc,\ acb = (bca)^{-1}\rangle, \quad \text{for } C \in \mathcal{M}_2, \end{cases}$$

where e is the unit element.

Since the topology of the pair $(\mathbb{CP}^2, C)$ is independent on the choice of C in $\mathcal{M}_1$ (respectively in $\mathcal{M}_2$) (cf. [23,24,7]), to prove our theorem it suffices to consider the fundamental groups $\pi_1(\mathbb{CP}^2 - C_1)$ and $\pi_1(\mathbb{CP}^2 - C_2)$. The latter are computed in Sections 2 and 3 below. We do not know if they are isomorphic or not (see Section 4 for some observations). Nevertheless, it follows from the general theory of the algebraic fundamental group (cf. [6,1,19,20]) that their profinite completions are isomorphic. Indeed, it is well known that the algebraic fundamental group is the profinite completion of the (topological) fundamental group, and on the other hand if $\hat{\iota}: \mathbb{C} \to \mathbb{C}$ is an extension of the Galois automorphism $\iota: \mathbb{Q}(\sqrt{3}) \to \mathbb{Q}(\sqrt{3})$ defined by $\iota(\sqrt{3}) = -\sqrt{3}$, then clearly $\hat{\iota}(C_1) = C_2$ so that the algebraic fundamental groups $\pi_1^{\mathrm{alg}}(\mathbb{CP}^2 - C_1)$ and $\pi_1^{\mathrm{alg}}(\mathbb{CP}^2 - C_2)$ are isomorphic (in particular the algebraic fundamental group $\pi_1^{\mathrm{alg}}(\mathbb{CP}^2 - C)$ does not depend on the choice of the curve C in $\mathcal{M}$). Also, we can observe that the abelianizations of the commutators subgroups of $\pi_1(\mathbb{CP}^2 - C_1)$ and $\pi_1(\mathbb{CP}^2 - C_2)$ are both isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$. The computation of the commutators subgroups can be done via Reidemeister–Schreier's method (cf. [8]) or can be directly checked using GAP4.

To compute $\pi_1(\mathbb{CP}^2 - C_1)$ and $\pi_1(\mathbb{CP}^2 - C_2)$, we use the classical Zariski–van Kampen theorem (cf. [22,21,3]) with the vertical pencil lines $L_\eta := \{x = \eta\}$, $\eta \in \mathbb{C}$ (for details on Zariski–van Kampen pencil method and the terminology we use hereafter, we refer to our previous works [4,5]). Both cases C_1 and C_2 have 5 singular pencil lines. Each monodromy associated to such a singular line induces a permutation of the 6 points of the intersection of a generic pencil line with the curve. The subgroups of $\mathfrak{S}_6$ generated by these permutations, for C_1 and C_2, are isomorphic and their order is 48 (here $\mathfrak{S}_6$ designates the group of permutations of 6 elements).

Independently of these observations, notice also that the generic Alexander polynomials of C_1 and C_2 are equal to $t^2 - t + 1$ (cf. [12]). So, if $\pi_1(\mathbb{CP}^2 - C_1)$ and $\pi_1(\mathbb{CP}^2 - C_2)$ were not isomorphic, the pairs of curves (C_1, C_2) would be the first example of Alexander-equivalent Zariski pair (see [2,10,11] for the definition) dealing with irreducible curves of degree 6. At present, the lowest known degree which gives Alexander-equivalent Zariski pairs for irreducible curves is 8 (cf. [11]). On the other hand, if $\pi_1(\mathbb{CP}^2 - C_1)$ and $\pi_1(\mathbb{CP}^2 - C_2)$ were isomorphic, then (C_1, C_2) might be the first candidate for a π_1-equivalent Zariski pair of irreducible curves (see [5] for the definition). But, although the curves C_1 and C_2 belong to different irreducible components of $\mathcal{M}$, it is a very hard problem to determine if the pairs of spaces $(\mathbb{CP}^2, C_1)$ and $(\mathbb{CP}^2, C_2)$ are homeomorphic or not.

Remark. Both of C_1 and C_2 are defined over $\mathbb{Q}(\sqrt{3})$ and their singular points are in $\mathbb{RP}^2$. We can easily see that $(\mathbb{RP}^2, C_1)$ and $(\mathbb{RP}^2, C_2)$ have different topologies.

2. Fundamental group of $\mathbb{CP}^2 - C_1$

The singularities E_6 of C_1 are situated at $(0, 1)$ and at $(0, -1)$, the singularity A_2 at $(-1, 0)$ and the singularity A_5 at $(1, 0)$. Fig. 1 shows the real plane section of the curve (in the figures, we do not respect the numerical scale).

As mentioned above, to compute the fundamental group, we use the classical Zariski–van Kampen theorem (cf. [22,21,3]) with the vertical pencil lines $L_\eta := \{x = \eta\}$, $\eta \in \mathbb{C}$. For more details on Zariski–van Kampen pencil method and the terminology we use hereafter, we refer to our previous works [4,5].

The discriminant $\Delta_y(f_1)$ of f_1 as a polynomial in y, which describes the singular lines of our pencil (notice that the line at infinity $\{Z = 0\}$ is not singular), is a polynomial in x given by

$$\begin{aligned}\Delta_y(f_1)(x) = {} & a_0(71x + 19 + 12\sqrt{3})(251x + 117 + 48\sqrt{3})^2 \\ & \times (x + 1)^3(x - 1)^8 x^{16},\end{aligned}$$

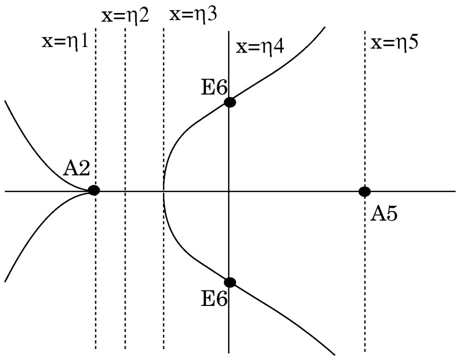

Fig. 1. Real plane section of C_1.

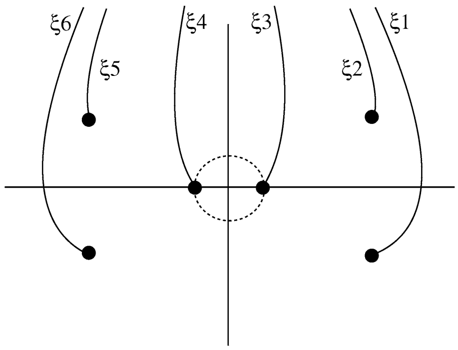

Fig. 2. Generators at $x = \eta_3 + \varepsilon$.

where a_0 is some real number. This polynomial has exactly five distinct roots which are all real numbers:

$$\eta_1 = -1, \qquad \eta_2 = -0.797\ldots, \qquad \eta_3 = -0.560\ldots, \qquad \eta_4 = 0, \qquad \eta_5 = 1.$$

The singular lines of the pencil are the lines $L_{\eta_1}, \ldots, L_{\eta_5}$ corresponding to these five roots.

We consider the generic line $L_{\eta_3+\varepsilon}$ and we take generators $\xi_1, \ldots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_3+\varepsilon} - C_1)$ as in Fig. 2. Everywhere in the text, ε is a sufficiently small strictly positive real number. The base point for the fundamental groups is always chosen to be the point $(0:1:0)$ which is the axis of our pencil and also equal to the point at infinity of $L_\eta \simeq \mathbb{CP}^1$. In the figures, for simplicity of drawing pictures, we denote a lasso oriented counter-clockwise (cf. [9]) just by a path ending with a bullet.

The monodromy relations around the singular lines of the pencil are given as follows.

Monodromy relations at $x = \eta_3$. The monodromy relation around the singular line $x = \eta_3$ (obtained when x runs once counter-clockwise on the circle $|x - \eta_3| = \varepsilon$) is just a tangent-type relation:

(r$_1$) $\quad \xi_3 = \xi_4.$

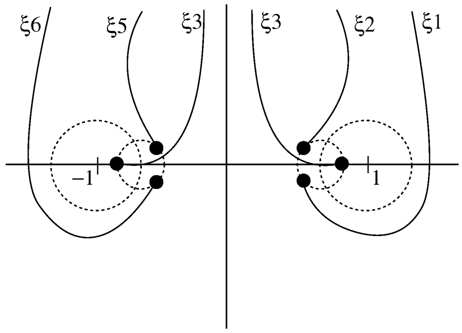

Fig. 3. Generators at $x = \eta_4 - \varepsilon$.

Monodromy relations at $x = \eta_4$. To read the monodromy relations around the singular line $x = \eta_4$ (obtained when x moves on the real axis from $x := \eta_3 + \varepsilon \to \eta_4 - \varepsilon$, then runs once counter-clockwise on the circle $|x - \eta_4| = \varepsilon$, and then comes back on the real axis from $x := \eta_4 - \varepsilon \to \eta_3 + \varepsilon$), we first show in Fig. 3 how the generators at $x = \eta_3 + \varepsilon$ (cf. Fig. 2) are deformed when x moves on the real axis from $x := \eta_3 + \varepsilon \to \eta_4 - \varepsilon$. Then, we look at the Puiseux parametrizations of C_1 at (0, 1):

$$\begin{cases} x = t^3, \\ y = 1 + (-3 + 2\sqrt{3})t^3 + (-3762 + 2172\sqrt{3})^{1/3}t^4 + \text{higher terms,} \end{cases}$$

and at (0, −1):

$$\begin{cases} x = t^3, \\ y = -1 + (3 - 2\sqrt{3})t^3 - (-3762 + 2172\sqrt{3})^{1/3}t^4 + \text{higher terms.} \end{cases}$$

The first system of equations shows that when $x = \varepsilon \exp(i\theta)$ moves around $\eta_4 = 0$ once counter-clockwise, the topological behavior of the three points near 1 in Fig. 3 looks like the movement of three satellites (corresponding to $t = \varepsilon^{1/3} \exp(i\nu)$, $\nu = \theta/3, \theta/3 + 2\pi/3, \theta/3 + 4\pi/3$) accompanying a planet. The movement of the planet is described by the term $(-3 + 2\sqrt{3})t^3$. The latter turns once counter-clockwise around 1. This movement can obviously be forgotten here. The movement of the satellites around the planet is described by the term $(-3762 + 2172\sqrt{3})^{1/3}t^4$. Each of them does (4/3)-turn counter-clockwise around the planet. So, the monodromy relations at $x = \eta_4$ coming from the singular point (0, 1) are given by

(r$_2$) $\xi_1 = (\xi_3\xi_2\xi_1)\xi_2^{-1}\xi_3\xi_2(\xi_3\xi_2\xi_1)^{-1}$,

(r$_3$) $\xi_2^{-1}\xi_3\xi_2 = (\xi_3\xi_2\xi_1)\xi_2(\xi_3\xi_2\xi_1)^{-1}$.

Similarly, using the second system of equations, one obtains that the monodromy relations at $x = \eta_4$ coming from the singular point (0, −1) are given by

(r$_4$) $\xi_5 = (\xi_6\xi_5\xi_3)\xi_5\xi_3\xi_5^{-1}(\xi_6\xi_5\xi_3)^{-1}$,

(r$_5$) $\xi_5\xi_3\xi_5^{-1} = (\xi_6\xi_5\xi_3)\xi_6(\xi_6\xi_5\xi_3)^{-1}$.

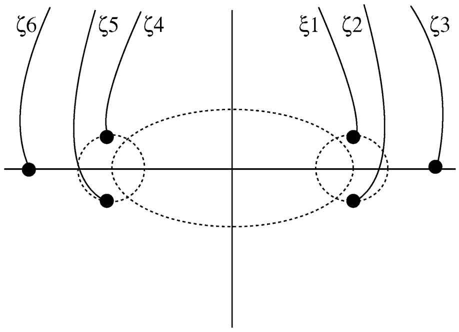

Fig. 4. Generators at $x = \eta_5 - \varepsilon$.

Monodromy relations at $x = \eta_5$. To read the monodromy relations at $x = \eta_5$, we first need to know how the generators at $x = \eta_4 - \varepsilon$ (cf. Fig. 3) are deformed when x does first a half-turn counter-clockwise on the circle $|x - \eta_4| = \varepsilon$ and then moves on the real axis from $x := \eta_4 + \varepsilon \to \eta_5 - \varepsilon$. This deformation is shown in Fig. 4, where

$$\begin{aligned}
&\zeta_2 := (\xi_2\xi_1)^{-1}\xi_3(\xi_2\xi_1),\\
&\zeta_3 := (\xi_3\xi_2\xi_1)^{-1}\xi_2(\xi_3\xi_2\xi_1),\\
&\zeta_4 := (\xi_6\xi_5\xi_3)^{-1}\xi_5\xi_3\xi_5^{-1}(\xi_6\xi_5\xi_3),\\
&\zeta_5 := (\xi_6\xi_5\xi_3)^{-1}(\xi_5\xi_3)\xi_5(\xi_5\xi_3)^{-1}(\xi_6\xi_5\xi_3),\\
&\zeta_6 := (\xi_5\xi_3)^{-1}\xi_6(\xi_5\xi_3).
\end{aligned}$$

Then, we look at the Puiseux parametrizations of the two branches K_1 and K_2 of C_1 at $(1, 0)$:

$$\begin{aligned}
&K_1: \quad x = 1 + t^2, \qquad y = c_1 \exp(i\pi/2)t + c_2 \exp(3i\pi/2)t^2 + \text{higher terms},\\
&K_2: \quad x = 1 + t^2, \qquad y = c_1 \exp(i\pi/2)t + c_2 \exp(i\pi/2)t^2 + \text{higher terms},
\end{aligned}$$

where c_1 and c_2 are strictly positive real numbers. And, as above, one deduces from these equations that the monodromy relations at $x = \eta_5$ are given by

$$\begin{aligned}
&(\mathrm{r}_6) \quad \xi_1 = (\xi_6\xi_5\xi_3)^{-1}(\xi_5\xi_3)\xi_5\xi_3\xi_5^{-1}(\xi_5\xi_3)^{-1}(\xi_6\xi_5\xi_3),\\
&(\mathrm{r}_7) \quad (\xi_2\xi_1)^{-1}\xi_3(\xi_2\xi_1) = (\xi_6\xi_5\xi_3)^{-1}(\xi_5\xi_3)\xi_5(\xi_5\xi_3)^{-1}(\xi_6\xi_5\xi_3),\\
&(\mathrm{r}_8) \quad \xi_3 = (\xi_6\xi_5\xi_3)\xi_2^{-1}(\xi_3\xi_2)\xi_1(\xi_3\xi_2)^{-1}\xi_2(\xi_6\xi_5\xi_3)^{-1}.
\end{aligned}$$

Monodromy relations at $x = \eta_2$. In Fig. 5, we show how the generators at $x = \eta_3 + \varepsilon$ (cf. Fig. 2) are deformed when x does first a half-turn counter-clockwise on the circle $|x - \eta_3| = \varepsilon$ and then moves on the real axis from $x := \eta_3 - \varepsilon \to \eta_2 + \varepsilon$. To see this deformation, we can observe that C_1 is the double covering of the curve $f_1(x, \sqrt{y}) = 0$ along $y = 0$ (cf. Fig. 6) and then we use the same argument as in [15]. The monodromy relations at $x = \eta_2$ are tangent-type relations:

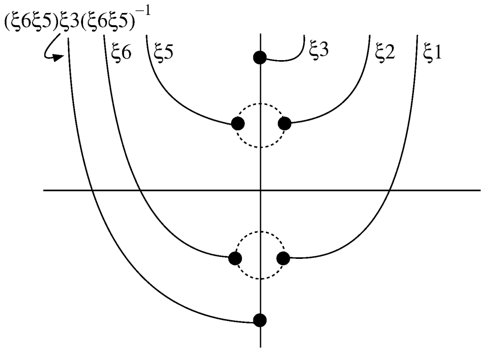

Fig. 5. Generators at $x = \eta_2 + \varepsilon$.

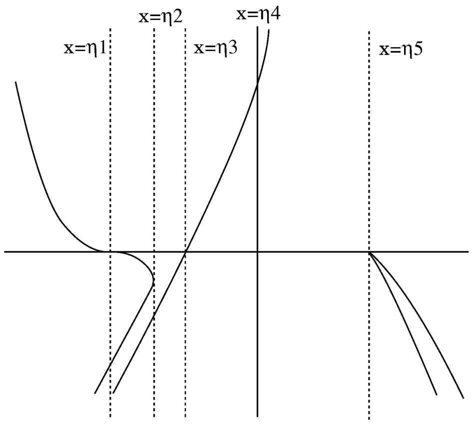

Fig. 6. Real plane section of $f_1(x, \sqrt{y}) = 0$.

(r$_9$) $\xi_2 = \xi_3^{-1}\xi_5\xi_3$,

(r$_{10}$) $\xi_1 = (\xi_5\xi_3\xi_2)^{-1}\xi_6(\xi_5\xi_3\xi_2)$.

Monodromy relations at $x = \eta_1$. When x does first a half-turn counter-clockwise on the circle $|x - \eta_2| = \varepsilon$ from $\eta_2 + \varepsilon$ and then moves on the real axis from $x := \eta_2 - \varepsilon \to \eta_1 + \varepsilon$, the generators at $x = \eta_2 + \varepsilon$ (cf. Fig. 5) are deformed as in Fig. 7. The Newton principal part of f_1 at $(-1, 0)$ is given by

$$(1248\sqrt{3} - 2160)y^2 + 16(3\sqrt{3} - 5)(x + 1)^3.$$

The monodromy relation at $x = \eta_1$ is given by the braid relation:

(r$_{11}$) $\xi_1 \cdot (\xi_3\xi_2)^{-1}\xi_5(\xi_3\xi_2) \cdot \xi_1 = (\xi_3\xi_2)^{-1}\xi_5(\xi_3\xi_2) \cdot \xi_1 \cdot (\xi_3\xi_2)^{-1}\xi_5(\xi_3\xi_2)$.

Vanishing relation at infinity. It is given by

(r$_{12}$) $\xi_6\xi_5\xi_4\xi_3\xi_2\xi_1 = e$.

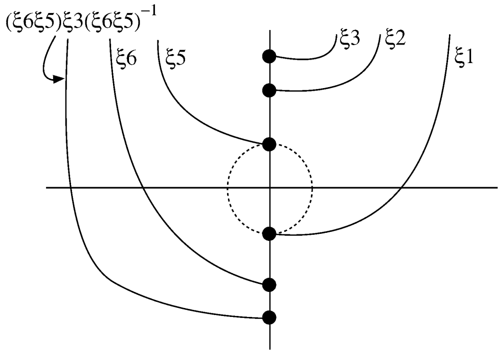

Fig. 7. Generators at $x = \eta_1 + \varepsilon$.

Simplification of the presentation. First, we write the relation (r_9) as

$$\xi_5 = \xi_3\xi_2\xi_3^{-1},$$

and we replace in (r_{10}). This gives

$$\xi_6 = (\xi_3\xi_2^2)\xi_1(\xi_3\xi_2^2)^{-1}.$$

Then, using (r_1), we write the relation (r_{12}) as $\xi_6\xi_5\xi_3 = (\xi_3\xi_2\xi_1)^{-1}$. The latter, combined with (r_9), shows that (r_4) can be written as

$$\xi_2 = (\xi_1\xi_3)^{-1}\xi_3(\xi_1\xi_3).$$

Also, combined with the relations (r_9), (r_{10}) and (r_4) (which give ξ_5, ξ_6 and ξ_2 respectively), it shows that (r_5) is written as $(\xi_3\xi_1)\xi_3(\xi_3\xi_1)^{-1} = \xi_2\xi_1\xi_2^{-1}$ and (r_6) as $\xi_1\xi_3\xi_1(\xi_3\xi_2)^2 = \xi_2\xi_1(\xi_3\xi_2)^2\xi_3$. Now, by ($r_9$), we can write ($r_{11}$) as $\xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2$. The latter, combined with the relation (r_4) (which gives ξ_2), shows that (r_2) is written as

$$(r_2') \quad (\xi_1\xi_3\xi_1)^2 = (\xi_3\xi_1\xi_3)^2.$$

Also, combined with (r_4) and (r_2'), it shows that (r_3) is equivalent to

$$(r_3') \quad \xi_3\xi_1\xi_3^3 = \xi_1^3\xi_3\xi_1.$$

The relation (r_{12}) can be also written in the generators ξ_1 and ξ_3 using the relations (r_{10}), (r_9) and (r_4) (which give ξ_6, ξ_5 and ξ_2 respectively) combined with the relations (r_3') and (r_2'). Precisely, (r_{12}) is equivalent to

$$(r_{12}') \quad (\xi_1\xi_3)^3 = e.$$

We have seen that (r_5) can be written as $(\xi_3\xi_1)\xi_3(\xi_3\xi_1)^{-1} = \xi_2\xi_1\xi_2^{-1}$. Thus, using the relation (r_{11}), under the form $\xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2$ (see above), and the relations (r_4) (which gives ξ_2) and (r_{12}'), we can write (r_5) as $(\xi_3\xi_1)^3 = e$ so that (r_5) is equivalent to (r_{12}') and can thus be omitted. We can also eliminate ξ_2 in the relation (r_6), written under the form $\xi_1\xi_3\xi_1(\xi_3\xi_2)^2 = \xi_2\xi_1(\xi_3\xi_2)^2\xi_3$, using the relation ($r_4$) (which gives ξ_2) and the relation (r_2'). Precisely, we obtain that (r_6) is equivalent to $\xi_3^3\xi_1\xi_3 = \xi_1\xi_3\xi_1^3$. And, again, the latter relation can be omitted as it is automatically satisfied by (r_3') and (r_{12}'). On the other hand, since $\xi_6\xi_5\xi_3 = (\xi_3\xi_2\xi_1)^{-1}$ (by (r_{12})), by combining (r_9) and (r_4) (which give ξ_5 and ξ_2 respectively) with (r'_{12}) and (r'_2), we show easily that (r_7) is always satisfied. Also, by combining (r_4) and (r_2) with (r'_3) and (r'_{12}), we can show that (r_8) is always satisfied too. Similarly, by combining (r_4) with (r'_{12}), (r'_3) and (r'_2), we can prove that the relation (r_{11}), that is $\xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2$ (see above), is also automatically satisfied.

Finally, we have proved that $\pi_1(\mathbb{CP}^2 - C_1)$ is presented by the generators ξ_1 and ξ_3 and the relations (r'_2), (r'_3) and (r'_{12}). If we put $a := \xi_1$ and $b := \xi_3$ we obtain the presentation announced in the theorem:

$$\pi_1\bigl(\mathbb{CP}^2 - C_1\bigr) = \bigl\langle a, b \mid (aba)^2 = (bab)^2,\ bab^3 = a^3ba,\ (ab)^3 = e\bigr\rangle.$$

3. Fundamental group of $\mathbb{CP}^2 - C_2$

The location of the singularities of C_2 is the same with that of C_1, namely: 2 E_6 at $(0, \pm 1)$, A_2 at $(-1, 0)$ and A_5 at $(1, 0)$. Fig. 8 shows the real plane section of the curve.

To compute the fundamental group, we apply again the Zariski–van Kampen theorem with the vertical pencil lines $L_\eta := \{x = \eta\}$, $\eta \in \mathbb{C}$.

The discriminant $\Delta_y(f_2)$ of f_2 as a polynomial in y is given by

$$\Delta_y(f_2)(x) = a_0(71x + 19 - 12\sqrt{3})(251x + 117 - 48\sqrt{3})^2(x + 1)^3(x - 1)^8x^{16},$$

where a_0 is some real number. This polynomial has also exactly five distinct roots which are all real numbers:

$$\eta_1 = -1, \qquad \eta_2 = -0.134\ldots, \qquad \eta_3 = 0, \qquad \eta_4 = 0.025\ldots, \qquad \eta_5 = 1.$$

The singular lines of the pencil are the lines $x = \eta_i$, for $1 \leqslant i \leqslant 5$, corresponding to these five roots (notice that the line at infinity is not singular).

We consider the generic line $L_{\eta_2+\varepsilon}$ and we take generators $\xi_1, \ldots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_2+\varepsilon} - C_2)$ as in Fig. 9.

The monodromy relations around the singular lines of the pencil are given as follows.

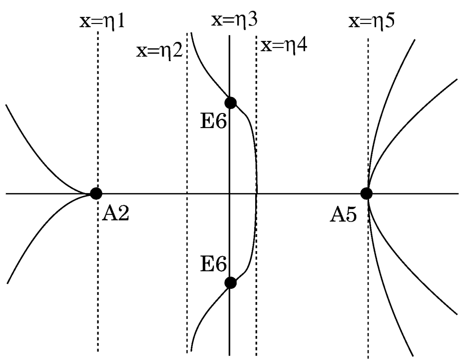

Fig. 8. Real plane section of C_2.

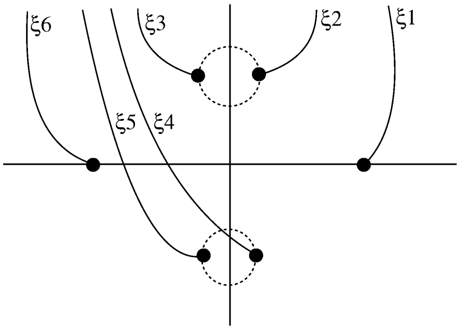

Fig. 9. Generators at $x = \eta_2 + \varepsilon$.

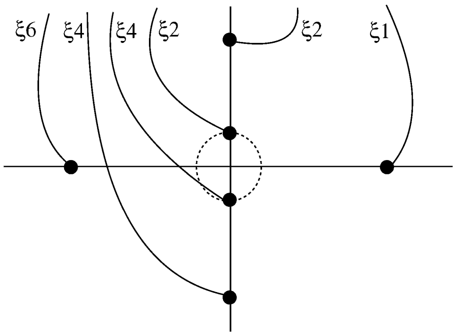

Fig. 10. Generators at $x = \eta_1 + \varepsilon$.

Monodromy relations at $x = \eta_2$. The monodromy relations around the singular line $x = \eta_2$ are just tangent-type relations:

(r$_1$) $\xi_2 = \xi_3$,

(r$_2$) $\xi_4 = \xi_5$.

Monodromy relations at $x = \eta_1$. In Fig. 10, we show how the generators at $x = \eta_2 + \varepsilon$ (cf. Fig. 9) are deformed when x does first a half-turn counter-clockwise on the circle $|x - \eta_2| = \varepsilon$ and then moves on the real axis from $x := \eta_2 - \varepsilon \to \eta_1 + \varepsilon$. As above, to see this deformation, we can observe that C_2 is the double covering of the curve $f_2(x, \sqrt{y}\,) = 0$ along $y = 0$ (cf. Fig. 11) and then we use the same argument as in [15]. The Newton principal part of f_2 at $(-1, 0)$ is given by

$$(-2160 - 1248\sqrt{3}\,)y^2 - 16(3\sqrt{3} + 5)(x + 1)^3.$$

The monodromy relation around the singular line $x = \eta_1$ is given by the braid relation

(r$_3$) $\xi_2\xi_4\xi_2 = \xi_4\xi_2\xi_4$.

Monodromy relations at $x = \eta_3$. When x moves on the real axis from $x := \eta_2 + \varepsilon \to \eta_3 - \varepsilon$, the generators at $x = \eta_2 + \varepsilon$ (cf. Fig. 9) are deformed as in Fig. 12. To read the

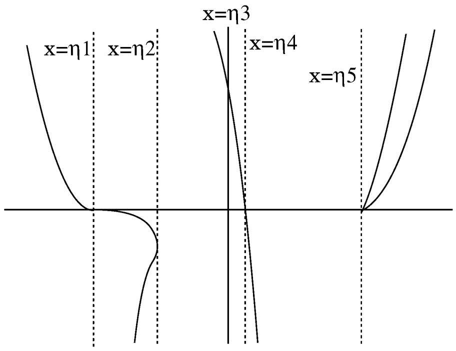

Fig. 11. Real plane section of $f_2(x, \sqrt{y}) = 0$.

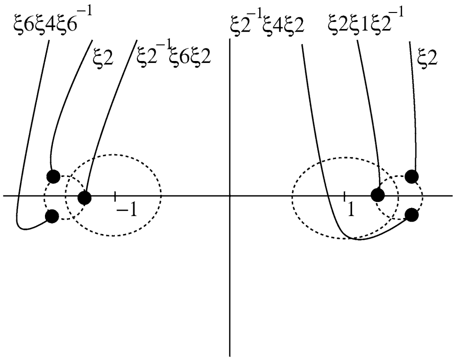

Fig. 12. Generators at $x = \eta_3 - \varepsilon$.

monodromy relations around the singular line $x = \eta_3$, we look at the Puiseux parametrizations of C_2 at $(0, 1)$:

$$\begin{cases} x = t^3, \\ y = 1 + (-3 - 2\sqrt{3})t^3 - (3762 + 2172\sqrt{3})^{1/3}t^4 + \text{higher terms,} \end{cases}$$

and at $(0, -1)$:

$$\begin{cases} x = t^3, \\ y = -1 + (3 + 2\sqrt{3})t^3 + (3762 + 2172\sqrt{3})^{1/3}t^4 + \text{higher terms.} \end{cases}$$

From these equations, one deduces that the monodromy relations at $x = \eta_3$ coming from the singular point $(0, 1)$ are given by

$$(\mathrm{r}_4) \quad \xi_2 = \big(\xi_4\xi_2^2\xi_1\big)\xi_2\xi_1\xi_2^{-1}\big(\xi_4\xi_2^2\xi_1\big)^{-1},$$

$$(\mathrm{r}_5) \quad \xi_2\xi_1\xi_2^{-1} = \big(\xi_2^{-1}\xi_4\xi_2^2\xi_1\big)\xi_2^{-1}\xi_4\xi_2\big(\xi_2^{-1}\xi_4\xi_2^2\xi_1\big)^{-1},$$

while those coming from $(0, -1)$ are given by

$$(\mathrm{r}_6) \quad \xi_2^{-1}\xi_6\xi_2 = (\xi_6\xi_4)\xi_2(\xi_6\xi_4)^{-1},$$

$$(\mathrm{r}_7) \quad \xi_2 = (\xi_6\xi_4\xi_2)\xi_6\xi_4\xi_6^{-1}(\xi_6\xi_4\xi_2)^{-1}.$$

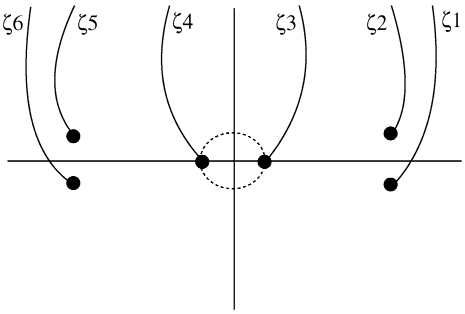

Fig. 13. Generators at $x = \eta_4 - \varepsilon$.

Monodromy relations at $x = \eta_4$. When x does first a half-turn counter-clockwise on the circle $|x - \eta_3| = \varepsilon$ from $\eta_3 - \varepsilon$ and then moves on the real axis from $x := \eta_3 + \varepsilon \to \eta_4 - \varepsilon$, the generators at $x = \eta_3 - \varepsilon$ (cf. Fig. 12) are deformed as in Fig. 13, where

$$\begin{aligned}
\zeta_1 &:= \big(\xi_4\xi_2^2\xi_1\big)^{-1}\xi_2\big(\xi_4\xi_2^2\xi_1\big),\\
\zeta_2 &:= \big(\xi_4\xi_2^2\xi_1\big)^{-1}\xi_2^2\xi_1\xi_2^{-2}\big(\xi_4\xi_2^2\xi_1\big),\\
\zeta_3 &:= (\xi_2\xi_1)^{-1}\xi_2^{-1}\xi_4\xi_2(\xi_2\xi_1),\\
\zeta_4 &:= (\xi_6\xi_4\xi_2)^{-1}\xi_2(\xi_6\xi_4\xi_2),\\
\zeta_5 &:= \xi_2^{-1}\xi_4\xi_2,\\
\zeta_6 &:= \xi_2^{-1}\xi_6\xi_2.
\end{aligned}$$

The monodromy relation at $x = \eta_4$ is a tangent-type relation:

$$(\mathrm{r}_8)\quad (\xi_2\xi_1)^{-1}\xi_2^{-1}\xi_4\xi_2(\xi_2\xi_1) = (\xi_6\xi_4\xi_2)^{-1}\xi_2(\xi_6\xi_4\xi_2).$$

Monodromy relations at $x = \eta_5$. When x does first a half-turn counter-clockwise on the circle $|x - \eta_4| = \varepsilon$ from $\eta_4 - \varepsilon$ and then moves on the real axis from $x := \eta_4 + \varepsilon \to \eta_5 - \varepsilon$, the generators at $x = \eta_4 - \varepsilon$ (cf. Fig. 13) are deformed as in Fig. 14. To read the monodromy relations at $x = \eta_5$, we look at the Puiseux parametrizations of the two branches K_1 and K_2 of C_2 at $(1, 0)$:

$$\begin{aligned}
K_1&:\quad x = 1 + t^2, \qquad y = c_1t - c_2t^2 + \text{higher terms},\\
K_2&:\quad x = 1 + t^2, \qquad y = c_1t + c_2t^2 + \text{higher terms},
\end{aligned}$$

where c_1 and c_2 are strictly positive real numbers. These equations show that the monodromy relations at $x = \eta_5$ are given by

$$\begin{aligned}
&(\mathrm{r}_9)\quad \big(\xi_2^2\xi_1\big)\xi_2^{-1}\xi_4\xi_2\big(\xi_2^2\xi_1\big)^{-1} = \xi_4^{-1}\xi_6\xi_4,\\
&(\mathrm{r}_{10})\quad \xi_2 = \big(\xi_1\xi_2^{-1}\xi_4\xi_2\big)\xi_1\big(\xi_1\xi_2^{-1}\xi_4\xi_2\big)^{-1},\\
&(\mathrm{r}_{11})\quad \xi_2\xi_1\xi_2^{-1} = (\xi_6\xi_4)\xi_2(\xi_6\xi_4)^{-1}.
\end{aligned}$$

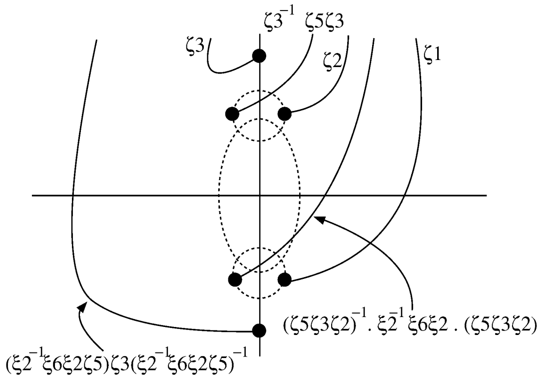

Fig. 14. Generators at $x = \eta_5 - \varepsilon$.

Vanishing relation at infinity. It is given by

$$(\mathrm{r}_{12}) \quad \xi_6\xi_4^2\xi_2^2\xi_1 = e.$$

Simplification of the presentation. By (r_{11}) and (r_{12}), we see that the relation (r_4) is always satisfied. Now, we write (r_6) as

$$(\mathrm{r}'_6) \quad \xi_6\xi_2\xi_6\xi_4 = \xi_2\xi_6\xi_4\xi_2.$$

The latter, combined with (r_{12}), shows that (r_{11}) is equivalent to

$$(\mathrm{r}'_{11}) \quad \xi_2\xi_6\xi_4 = (\xi_4\xi_6\xi_2)^{-1}.$$

On the other hand, by (r_{12}), we can write the relation (r_9) as

$$(\mathrm{r}'_9) \quad \xi_4\xi_2\xi_6\xi_4 = \xi_2\xi_6\xi_4\xi_6.$$

The latter, combined with (r_{12}), (r_3) and (r'_6), shows that (r_{10}) is equivalent to (r'_{11}). Also, combined with (r'_6), it shows that (r_7) is automatically satisfied. By (r_{12}) and (r'_6), one shows easily that (r_8) is equivalent to (r'_9). By (r_9), we can write (r_5) as $\xi_2\xi_1\xi_2^{-1} = (\xi_2^{-1}\xi_4)\xi_4^{-1}\xi_6\xi_4(\xi_2^{-1}\xi_4)^{-1}$, that is $\xi_2^2\xi_1\xi_2^{-2} = \xi_6$. Then, (r_{12}) shows that (r_5) is equivalent to (r'_{11}).

Finally, we have proved that $\pi_1(\mathbb{CP}^2 - C_2)$ is presented by the generators ξ_2, ξ_4 and ξ_6 and the relations (r_3), (r'_6), (r'_9) and (r'_{11}). If we put $a := \xi_2$, $b := \xi_4$ and $c := \xi_6$ we obtain the presentation announced in the theorem:

$$\pi_1(\mathbb{CP}^2 - C_2) = \langle a, b, c \mid aba = bab,\ cacb = acba,\\ bacb = acbc, acb = (bca)^{-1}\rangle.$$

4. Remarks

Consider the one-parameter family of non-tame torus sextics $(C(t))_{t\in\mathbb{C}}$, where $C(t)$ is given by the affine equation $g_t(x, y) = 0$ with

$$
\begin{aligned}
g_t(x,y) = {} & \bigl(-ty^2+(-1-t)x^2+x+t\bigr)^3 \\
& + \left(\left(\frac{1}{4}\sqrt{2}-\frac{3}{2}\sqrt{2}t\right)xy^2+\left(-\frac{3}{4}\sqrt{2}-\frac{3}{2}\sqrt{2}t\right)x^3\right. \\
& \left. + \sqrt{2}x^2+\left(-\frac{1}{4}\sqrt{2}+\frac{3}{2}\sqrt{2}t\right)x\right)^2.
\end{aligned}
$$

All the curves in this family are irreducible, except $C(0)$ and $C(1/2)$, and all of them have the configuration of singularities $\{A_1+A_5+2E_6\}$ (2 E_6 at $(0,\pm 1)$, A_5 at $(1,0)$, A_1 at $(-1,0)$), except $C(t_1)$ and $C(t_2)$ where $t_1=(3+2\sqrt{3})/6$ and $t_2=(3-2\sqrt{3})/6$. The curves $C(t_1)$ and $C(t_2)$ correspond to our sextics C_1 and C_2 respectively.

Proposition. *For any parameter $t\in\mathbb{C}-\{0,1/2,t_1,t_2\}$, the fundamental group $\pi_1(\mathbb{CP}^2-C(t))$ is isomorphic to the free product $\mathbb{Z}_2*\mathbb{Z}_3$.*

Proof. It suffices to consider the case of $C(1)$. And in fact the calculation in this case can be almost entirely deduced from those of C_1 done in Section 2 (one must not confuse C_1 with $C(1)$!). The real plane section of $C(1)$ is given in Fig. 15. As above, we use Zariski–van Kampen's theorem with the vertical pencil lines $L_\eta := \{x=\eta\}$, $\eta\in\mathbb{C}$. This pencil has 6 singular lines $x=\eta_{1,1}$, $\eta_{1,2}$ and η_i, $2\leqslant i\leqslant 5$, with respect to the curve $C(1)$, where

$$
\begin{aligned}
&\eta_{1,1}=-1, \qquad \eta_{1,2}=-0.872\ldots, \qquad \eta_2=-0.760\ldots,\\
&\eta_3=-0.539\ldots, \qquad \eta_4=0, \qquad \eta_5=1.
\end{aligned}
$$

We observe that $\eta_{1,1}$, $\eta_{1,2}$ are bifurcated from the value η_1 corresponding to the curve C_1 and the others η_i, $2\leqslant i\leqslant 5$, correspond to those of C_1 when t moves on the real axis from $t:=t_1\to 1$.

We choose generators of $\pi_1(L_{\eta_3+\varepsilon}-C(1))$ as in Fig. 16 (in the case of the curve C_1, this situation corresponds to Fig. 2). The monodromy relations around the singular lines $x=\eta_i$, $2\leqslant i\leqslant 5$, are the same that the relations (r_j), $1\leqslant j\leqslant 10$, corresponding to the curve C_1 (cf. Section 2). The monodromy relation around $x=\eta_{1,2}$ is given by

(R) $\quad \xi_1=(\xi_3\xi_2)^{-1}\xi_5(\xi_3\xi_2)$.

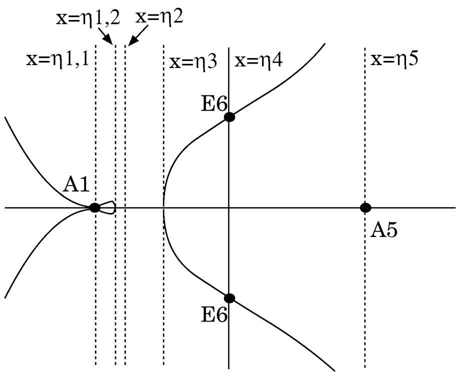

Fig. 15. Real plane section of $C(1)$.

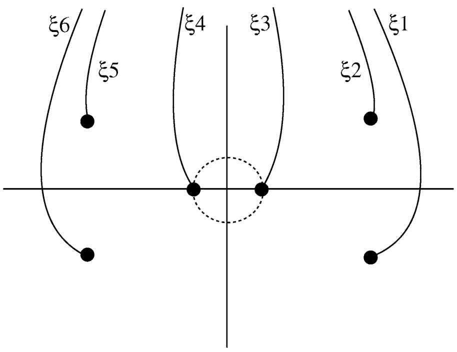

Fig. 16. Generators at $x = \eta_3 + \varepsilon$.

The latter shows that the monodromy relation around $x = \eta_{1,1}$ is automatically satisfied. The vanishing relation at infinity is the same that the relation (r_{12}) corresponding to the curve C_1 (cf. Section 2). Using (r_9), we can write (R) as $\xi_1 = \xi_2$. It is then a simple exercise (proceed as in the part entitled "Simplification of the presentation" in Section 2) to see that $\pi_1(\mathbb{CP}^2 - C(1))$ is presented by the generators ξ_1 and ξ_3 and the relations $\xi_1\xi_3\xi_1 = \xi_3\xi_1\xi_3$ and $(\xi_1\xi_3)^3 = e$, that is $\pi_1(\mathbb{CP}^2 - C(1)) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$. $\square$

Remark. Notice that our proof immediately implies that there is a surjective homomorphism

$$\pi_1\big(\mathbb{CP}^2 - C_1\big) \to \pi_1\big(\mathbb{CP}^2 - C(1)\big) \simeq \mathbb{Z}_2 * \mathbb{Z}_3.$$

There is also a surjective homomorphism $\pi_1(\mathbb{CP}^2 - C_2) \to \mathbb{Z}_2 * \mathbb{Z}_3$ in the case of the curve C_2 (this can be proved exactly by the same argument). In fact, since the family $(C(t))_{t\in\mathbb{C}}$ is a degeneration family at $t = t_1$ and $t = t_2$, such epimorphisms can be also deduced from the general theory of degeneration families (cf. [14, Theorem 8]).

Now, let K_i be the monodromy subgroup of the free group $\pi_1(L_{\tau_i} - C_i)$, where τ_i is chosen as $\eta_3 + \varepsilon$ for $i = 1$ (with η_3 from Section 2, i.e. $\eta_3 = -0.560\ldots$) and $\eta_2 + \varepsilon$ for $i = 2$ (with η_2 from Section 3, i.e. $\eta_2 = -0.134\ldots$). We assume that ε is small enough so that L_{τ_1} (respectively L_{τ_2}) is generic for any $C(t)$, $1 \leqslant t \leqslant t_1$ (respectively $C(t)$, $t_2 \leqslant t \leqslant t_2/2$). Let K_i' be the monodromy subgroup of $\pi_1(L_{\tau_i} - C(t))$ for $t = 1$ or $t_2/2$ according to $i = 1$ or 2. As we can identify $\pi_1(L_{\tau_i} - C_i)$ with $\pi_1(L_{\tau_i} - C(1))$ for $i = 1$ and with $\pi_1(L_{\tau_i} - C(t_2/2))$ for $i = 2$, we have canonical inclusions $K_i \subset K_i'$ (cf. remark above). On the other hand, there is a canonical identification isomorphism $\psi : \pi_1(L_{\tau_1} - C(1)) \xrightarrow{\sim} \pi_1(L_{\tau_2} - C(t_2/2))$ such that $\psi(K_1') = K_2'$. In fact, $\pi_1(L_{\tau_1} - C(t_2/2))$ and $\pi_1(L_{\tau_2} - C(t_2/2))$ are canonically identified by the pencil deformation (we assume that L_{τ_1} is also generic for $C(t_2/2)$). The identification of $\pi_1(L_{\tau_1} - C(1))$ and $\pi_1(L_{\tau_1} - C(t_2/2))$ is obtained by considering the following deformation inside the one-parameter family $C(t)$: we start from $t = 1$, then we move the parameter t as follows: on the real axis from $t := 1 \to 1/2 + \varepsilon$; half-turn counter-clockwise on the circle $|t - 1/2| = \varepsilon$; on the real axis from $t := 1/2 - \varepsilon \to \varepsilon$; half-turn counter-clockwise on the circle $|t| = \varepsilon$; and finally on the real axis from $t := -\varepsilon \to t_2/2$

(we assume that L_{τ_1} is also generic for every curve $C(t)$ in this deformation). In Sections 2 and 3 above, in order to make the computations of the fundamental groups $\pi_1(\mathbb{CP}^2 - C_1)$ and $\pi_1(\mathbb{CP}^2 - C_2)$ easier, we have chosen, in each case C_1 and C_2, a convenient generic line L_{τ_i} of the pencil and a convenient system of generators of $\pi_1(L_{\tau_i} - C_i)$. With such an arbitrary choice, it may be difficult *a priori* to compare "geometrically" the presentations of $\pi_1(\mathbb{CP}^2 - C_1)$ and $\pi_1(\mathbb{CP}^2 - C_2)$. However an explicit computation of $\psi : \pi_1(L_{\tau_1} - C_1) \to \pi_1(L_{\tau_2} - C_2)$ seems to show that $\psi(K_1) \neq K_2$. So, we doubt that $\pi_1(\mathbb{CP}^2 - C_1)$ and $\pi_1(\mathbb{CP}^2 - C_2)$ are (at least geometrically) isomorphic.

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References

[1] S. Abhyankar, Ramification Theoretic Methods in Algebraic Geometry, Ann. of Math. Stud., vol. 43, Princeton University Press, Princeton, NJ, 1959.
[2] E. Artal Bartolo, J. Carmona Ruber, Zariski pairs, fundamental groups and Alexander polynomials, J. Math. Soc. Japan 50 (1998) 521–543.
[3] D. Chéniot, Une démonstration du théorème de Zariski sur les sections hyperplanes d'une hypersurface projective et du théorème de van Kampen sur le groupe fondamental du complémentaire d'une courbe projective plane, Compositio Math. 27 (1973) 141–158.
[4] C. Eyral, M. Oka, On the fundamental groups of the complements of plane singular sextics, J. Math. Soc. Japan 57 (2005) 37–54.
[5] C. Eyral, M. Oka, π_1-equivalent weak Zariski pairs, Tokyo J. Math., in press.
[6] A. Grothendieck, et al., Revêtements étales et groupe fondamental (SGA 1), Lecture Notes in Math., vol. 224, Springer, Berlin, 1971.
[7] D.T. Lê, C.P. Ramanujam, The invariance of Milnor number implies the invariance of the topological type, Amer. J. Math. 98 (1976) 67–78.
[8] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations, second revised ed., Dover, New York, 1976.
[9] M. Oka, Two transforms of plane curves and their fundamental groups, J. Math. Sci. Univ. Tokyo 3 (1996) 399–443.
[10] M. Oka, Flex curves and their applications, Geom. Dedicata 75 (1999) 67–100.
[11] M. Oka, A new Alexander-equivalent Zariski pair, Acta Math. Vietnam. 27 (3) (2002) 349–357.
[12] M. Oka, Alexander polynomial of sextics, J. Knot Theory Ramifications 12 (5) (2003) 619–636.
[13] M. Oka, Geometry of reduced sextics of torus type, Tokyo J. Math. 26 (2) (2003) 301–327.
[14] M. Oka, A survey on Alexander polynomials of plane curves, in: Singularités Franco–Japonaises, CIRM, 2002, in: Séminaires et Congrès, vol. 10, Soc. Math. France, Paris, 2005, pp. 209–232.
[15] M. Oka, Symmetric plane curves with nodes and cusps, J. Math. Soc. Japan 44 (1992) 375–414.
[16] M. Oka, D.T. Pho, Fundamental group of sectics of torus type, in: Trends in Singularities, in: Trends in Mathematics, Birkhäuser, Basel, 2001, pp. 151–180.
[17] M. Oka, D.T. Pho, Classification of sextics of torus type, Tokyo J. Math. 25 (2) (2002) 399–433.
[18] D.T. Pho, Classification of singularities on torus curves of type (2, 3), Kodai Math. J. 24 (2001) 259–284.
[19] H. Popp, Fundamentalgruppen algebraischer Mannigfaltigkeiten, Lecture Notes in Math., vol. 176, Springer, Berlin, 1970.

[20] H. Popp, Introduction to the algebraic theory of the fundamental groups, in: Singularités à Cargèse, Rencontre Singularités Géom. Anal., Inst. Études Sci. de Cargèse 1972, in: Asterisque, vols. 7 and 8, Soc. Math. France, Paris, 1973, pp. 223–240.

[21] E.R. van Kampen, On the fundamental group of an algebraic curve, Amer. J. Math. 55 (1933) 255–260.

[22] O. Zariski, On the problem of existence of algebraic functions of two variables possessing a given branch curve, Amer. J. Math. 51 (1929) 305–328.

[23] O. Zariski, Studies in equisingularity II. Equisingularity in codimension 1 (and characteristic zero), Amer. J. Math. 87 (1965) 972–1006.

[24] O. Zariski, Contribution to the problem of equisingularity, in: Questions on Algebraic Varieties, CIME, III Ciclo, Varenna, 1969, Cremonese, Roma, 1970, pp. 261–343.